

ON SOME MOMENT MAPS AND INDUCED HOPF BUNDLES IN THE QUATERNIONIC PROJECTIVE SPACE

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ABSTRACT. We describe a diagram containing the zero sets of the moment maps associated to the diagonal $U(1)$ and $Sp(1)$ actions on the quaternionic projective space $\mathbb{H}P^n$. These sets are related both to focal sets of submanifolds and to Sasakian-Einstein structures on induced Hopf bundles. As an application, we construct a complex structure on the Stiefel manifolds $V_2(\mathbb{C}^{n+1})$ and $\tilde{V}_4(\mathbb{R}^{n+1})$, the one on the former manifold not being compatible with its known hypercomplex structure.

1. INTRODUCTION

The main motivation for the present work is to understand some aspects of the Riemannian geometry of the focal set $Foc_{\mathbb{H}P^n}\mathbb{C}P^n$, i.e. of the set of points in the quaternionic projective space $\mathbb{H}P^n$ that are critical values of the normal exponential map with respect to the totally geodesic submanifold $\mathbb{C}P^n \subset \mathbb{H}P^n$. Our starting point is the Habilitationsschrift of J. Berndt [7], [8], where in particular is proved that $Foc_{\mathbb{H}P^n}\mathbb{C}P^n$ fibers in circles over $Gr_2(\mathbb{C}^{n+1})$, the Grassmannian of 2-planes in \mathbb{C}^{n+1} . In the simplest cases $n = 1, 2$, these focal points distribute on spheres S^1, S^5 , respectively and $n = 3$ thus seems to be the first significant case. Indeed, since $Foc_{\mathbb{H}P^3}\mathbb{C}P^3$ is a circle bundle over the Klein quadric $Gr_2(\mathbb{C}^4) \cong \mathbb{C}Q^4 \subset \mathbb{C}P^5$, both the classical projective geometry of $\mathbb{C}Q^4$ (described for example in [39], pp. 26-35) and its two quaternion Kähler structures [27] can be related to the total space $Foc_{\mathbb{H}P^3}\mathbb{C}P^3$. More generally, the twofold Kähler-Einstein and quaternion Kähler properties of the Grassmannian $Gr_2(\mathbb{C}^{n+1})$ suggested to us a link between the focal set $Foc_{\mathbb{H}P^n}\mathbb{C}P^n$ and the Sasakian-Einstein geometry of the induced Hopf bundles considered in our previous work [31].

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This paper begins by giving an explicit identification of the focal set $Foc_{\mathbb{H}P^n} \mathbb{C}P^n$ with the zero level set of the moment map μ associated to the diagonal $U(1)$ - action on $\mathbb{H}P^n$. This is the simplest isometric action giving examples of reductions in quaternion Kähler geometry, and it is well-known that the reduced manifold is precisely the quaternion Kähler Wolf space $Gr_2(\mathbb{C}^{n+1})$ [20], [21]. According to the notation in use, we write $\mathbb{H}P^n // U(1) = Gr_2(\mathbb{C}^{n+1})$ to indicate this reduction procedure; this has been proved to be the unique one with respect to isometric circle actions and positive quaternion Kähler manifolds [3], [4]. Next, the diagonal $Sp(1)$ -action on $\mathbb{H}P^n$ has as reduction the quaternion Kähler manifold $\mathbb{H}P^n // Sp(1) = \tilde{Gr}_4(\mathbb{R}^{n+1})$, the Grassmannian of oriented 4-planes in \mathbb{R}^{n+1} . The zero set of the corresponding moment map $\nu : \mathbb{H}P^n \rightarrow \mathbb{H}$ can also be related to focal sets in $\mathbb{H}P^n$. (For these identifications cf. Theorems 3.1, 3.2).

On the other hand, the zero level sets $\mu^{-1}(0)$ and $\nu^{-1}(0)$ of these moment maps μ and ν can be identified with the total spaces of some induced Hopf S^1 -bundles. This enables us to define on them a Sasakian-Einstein and a 3-Sasakian metric, respectively, and hence to find a diagram where these zero sets $\mu^{-1}(0)$ and $\nu^{-1}(0)$ are base spaces of S^3 -bundles, now induced by the Hopf fibration $S^{4n+3} \rightarrow \mathbb{H}P^n$. The corresponding total spaces are the Stiefel manifolds $V_2(\mathbb{C}^{n+1})$ and $\tilde{V}_4(\mathbb{R}^{n+1})$, respectively. The 3-Sasakian structure of the fibers S^3 and the Sasakian structure of the base spaces $\mu^{-1}(0)$ and $\nu^{-1}(0)$ allow to construct a complex structure on $V_2(\mathbb{C}^{n+1})$ and on $\tilde{V}_4(\mathbb{R}^{n+1})$, whose definition is very much in the Calabi-Eckmann spirit. On the former Stiefel manifold, this complex structure is not compatible with its standard hypercomplex structure, obtained in [2], [14], [24]. On the other hand, both $V_2(\mathbb{C}^{n+1})$ and $\tilde{V}_4(\mathbb{R}^{n+1})$ are total spaces of framed bundles in Hopf surfaces over Kähler Einstein manifolds. Then these complex structures can be seen to belong to one-parameter families, also suitable for some exceptional cases, as shown in a forthcoming paper, [32].

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2. PRELIMINARIES

We collect here some definitions and basic facts that will be used throughout the paper.

Let (M, g) be a Riemannian manifold and let N be an isometrically immersed submanifold. The critical values of the restriction $\exp_{T^\perp N}$ of the exponential map of M to the normal bundle of N are called *focal points of N* and the set of all focal points, here denoted by $Foc_M N$, is called *the focal set of N* (see for example [18] p. 23, [19], p. 227, [28] p. 283). The focal set of a submanifold may not be a submanifold: for smooth plane curves and for regular surfaces in \mathbb{R}^3 the focal sets are respectively the evolutes and the surfaces of the centres, and both of them can have singular points, cf. [19], pp. 237-238 and p. 232. However, for the examples considered in the present paper all the focal sets turn out to be smooth. Indeed, it seems that not many focal sets of Riemannian submanifolds have been explicitly determined, but we can mention totally geodesic spheres in spheres [33], and hypersurfaces in space forms [16], [17].

We are mainly interested in $Foc_{\mathbb{H}P^n} \mathbb{C}P^n$, the focal set of the totally geodesic $\mathbb{C}P^n$ in the quaternionic projective space $\mathbb{H}P^n$. This focal set appears in J. Berndt's work [7], [8], denoted there by Q^n , and studied in connection with both the complex Kähler and the quaternion Kähler structure of $Gr_2(\mathbb{C}^{n+1})$, the Grassmannian of complex two-planes in \mathbb{C}^{n+1} .

Two geometric structures which appear naturally in our context are the Sasakian and the 3-Sasakian ones. We briefly recall the definitions, referring the reader to the survey [11] for further information.

A *Sasakian* manifold is a $(2n+1)$ -dimensional Riemannian manifold (S, g) equipped with a unitary Killing vector field ξ such that the field of endomorphisms $\varphi := \nabla \xi$ satisfies the differential equation

$$\nabla \varphi = Id \otimes \eta - g \otimes \xi,$$

where ∇ is the Levi-Civita connection of g and η is the dual 1-form of ξ . The data of a Sasakian structure on the manifold S is easily seen to be equivalent to the requirement that the cone metric $dr^2 + r^2 g$, on $\mathbb{R}_+ \times S$ have holonomy contained in $U(n+1)$. Note that η is a contact form on S , hence ξ is its Reeb field. In the simplest example, the Euclidean sphere S^{2n+1} , the Killing vector field is $\xi = -JU$, J being the standard complex structure of \mathbb{C}^{n+1} and U the unit outward normal to the sphere.

More generally, we look at the induced Hopf S^1 -bundle $\pi : V \rightarrow M$, over a smooth submanifold of $\mathbb{C}P^N$. Its total space V carries a Sasakian structure (V, g, ξ) induced from the one of (S^{2N+1}, can) . If

the Fubini-Study metric of $\mathbb{C}P^N$ induces an Einstein metric h with Einstein constant α on M , it can be seen (cf. [31], lemma 1) that the Ricci tensor of the metric g has the form

$$\text{Ric}(g) = \lambda g + \mu \eta \otimes \eta,$$

with $\lambda = \alpha - 2$, $\mu = \dim V + 1 - \alpha$. This is known as the η -Einstein property in Sasakian geometry and, following S. Tanno [38], an η -Einstein Sasakian metric can be deformed to a Sasakian-Einstein one by setting

$$(2.1) \quad g' = Ag + A(A - 1)\eta \otimes \eta,$$

so that $\xi' = A^{-1}\xi$ with $A = \frac{\lambda+2}{\dim V+1}$. Now π is a Riemannian submersion with respect to the metrics h and g ; thus to have a Riemannian submersion with respect to g' , we must consider the scaled metric $h' = Ah$ on the base M .

As Sasakian geometry may be viewed as the odd-dimensional counterpart of Kähler geometry, the odd-dimensional counterparts of hyperKähler manifolds are 3-Sasakian ones (cf. [11]). More precisely, a $(4n + 3)$ -dimensional Riemannian manifold (S, g) is said to be 3-Sasakian if it is endowed with three mutually orthogonal unit Killing vector fields ξ_1, ξ_2, ξ_3 , each one defining a Sasakian structure and satisfying the conditions:

$$[\xi_1, \xi_2] = 2\xi_3, \quad [\xi_2, \xi_3] = 2\xi_1, \quad [\xi_3, \xi_1] = 2\xi_2.$$

As above, an equivalent definition requires that the cone metric $dr^2 + r^2g$ on $\mathbb{R}_+ \times S$ be hyperkähler, i.e. its holonomy group be contained in $Sp(n + 1)$. 3-Sasakian manifolds are necessarily Einstein with positive scalar curvature and their Einstein constant is $4n + 2$. Given a positive quaternion Kähler manifold P , one constructs its Kähler-Einstein twistor space Z_P and then an S^1 bundle over it whose Chern class is, up to torsion, that of an induced Hopf bundle. The total space S thus obtained is an $SO(3)$ -principal bundle over P with 3-Sasakian structure. Moreover, all three fibrations involved are Riemannian submersions.

We recall now two basic moment maps of quaternion Kähler geometry. Let $[h_0 : h_1 : \dots : h_n]$ be the homogeneous coordinates on $\mathbb{H}P^n$: for each $a = 0, 1, \dots, n$, we shall write the complex and real components of h_a as follows:

$$(2.2) \quad h_a = z_a + w_a j = \alpha_a + \beta_a i + \gamma_a j + \delta_a k$$

where $z_a = \alpha_a + \beta_a i$, and $w_a = \gamma_a + \delta_a i$. The first moment map, induced by the diagonal action of $U(1)$ on $\mathbb{H}P^n$ is at the hyperkähler level of

the bundle $\mathbb{H}^{n+1} \setminus \{0\} \rightarrow \mathbb{H}P^n$, given by:

$$(2.3) \quad \mu : \mathbb{H}^{n+1} \setminus \{0\} \rightarrow \mathbb{H}, \quad \mu = \sum_{a=0}^n \bar{h}_a i h_a.$$

On $\mathbb{H}P^n$, one has to regard the corresponding moment map μ_{qK} as a section of S^2H , the rank 3 vector bundle of the compatible almost complex structures. In fact, by using the metric, μ_{qK} appears as a 2-form solution of

$$d\mu_{qK} = i_X \Omega,$$

where Ω is the Kähler 4-form and X the Killing vector field generating the $U(1)$ -action: [20], [1]. However, since the zero level sets $\mu^{-1}(0) \subset \mathbb{H}^{n+1}$ and $\mu_{qK}^{-1}(0) \subset \mathbb{H}P^n$ correspond to each other in the bundle projection $\mathbb{H}^{n+1} \setminus \{0\} \rightarrow \mathbb{H}P^n$, we shall refer to the definition $\mu = \sum_{a=0}^n \bar{h}_a i h_a$, whose zero set makes sense also when the h_a are the homogeneous coordinates of $\mathbb{H}P^n$ [11]. The reduced manifold $\mu_{qK}^{-1}(0)/U(1)$ turns out to be the quaternion Kähler Wolf space $\frac{SU(n+1)}{S(U(n-1) \times U(2))} \cong Gr_2(\mathbb{C}^{n+1})$ [20], [21].

The second moment map to be considered, generated by the action of $Sp(1)$, is (again at the hyperkähler level):

$$(2.4) \quad \nu : \mathbb{H}^{n+1} \setminus \{0\} \rightarrow \mathbb{H}^3, \quad \nu = \left(\sum_{a=0}^n \bar{h}_a i h_a, \sum_{a=0}^n \bar{h}_a j h_a, \sum_{a=0}^n \bar{h}_a k h_a \right),$$

and its corresponding quaternion Kähler moment map ν_{qK} can be viewed as a triple of 2-forms associated to the frame of Killing vector fields defining the $Sp(1)$ -action. The corresponding reduced manifold $\nu_{qK}^{-1}(0)/Sp(1)$ is now the Wolf space $\frac{SO(n+1)}{SO(n-3) \times SO(4)} \cong \widetilde{Gr}_4(\mathbb{R}^{n+1})$, the Grassmannian of oriented four-planes of \mathbb{R}^{n+1} .

3. STATEMENT OF RESULTS

Theorem 3.1. (i) *The focal set $Foc_{\mathbb{H}P^n} \mathbb{C}P^n$ coincides with $\mu^{-1}(0)$, and it is isometric to the total space of the induced Hopf S^1 -bundle via the Plücker embedding $Gr_2(\mathbb{C}^{n+1}) \hookrightarrow \mathbb{C}P^N$.*

(ii) *The metric g_1 , induced on $\mu^{-1}(0)$ by the Plücker embedding allows us to define on $\mu^{-1}(0)$ a Sasakian Einstein metric g .*

(iii) *The Stiefel manifold $V_2(\mathbb{C}^{n+1})$ of orthonormal 2-frames in \mathbb{C}^{n+1} , diffeomorphic to the total space of the induced Hopf S^3 -bundle via the embedding $\mu^{-1}(0) \subset \mathbb{H}P^n$, admits an (integrable) complex structure J , not compatible with the standard hypercomplex structure of $V_2(\mathbb{C}^{n+1})$.*

We have a similar statement regarding the moment map ν induced by the action of $Sp(1)$ on $\mathbb{H}P^n$. We need to consider the following mutually congruent, totally geodesic embeddings of $\mathbb{C}P^n$ in $\mathbb{H}P^n$.

$$(3.1) \quad \begin{aligned} \mathbb{C}P_i^n &= \{h \in \mathbb{H}P^n ; \gamma_a = \delta_a = 0, a = 1, \dots, n\}, \\ \mathbb{C}P_j^n &= \{h \in \mathbb{H}P^n ; \beta_a = \delta_a = 0, a = 1, \dots, n\}, \\ \mathbb{C}P_k^n &= \{h \in \mathbb{H}P^n ; \beta_a = \gamma_a = 0, a = 1, \dots, n\} : \end{aligned}$$

here $\mathbb{C}P_i^n$ is the standard $\mathbb{C}P^n$ appearing in Theorem 3.1.

Theorem 3.2. (i) *The zero level set $\nu^{-1}(0)$ coincides with the intersection $M = \text{Foc}_{\mathbb{H}P^n} \mathbb{C}P_i^n \cap \text{Foc}_{\mathbb{H}P^n} \mathbb{C}P_j^n \cap \text{Foc}_{\mathbb{H}P^n} \mathbb{C}P_k^n$ and is isometric to the total space of the induced Hopf S^1 -bundle over the Fano manifold $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$, via the embeddings $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})} \hookrightarrow Gr_2(\mathbb{C}^{n+1}) \hookrightarrow \mathbb{C}P^N$, the first of which is defined by regarding $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$ as the space of totally isotropic two-planes in \mathbb{C}^{n+1} . This isometry allows the construction of a Sasakian Einstein metric on $\nu^{-1}(0)$, and identifies it with the homogeneous 3-Sasakian manifold $SO(n+1)/(SO(n-3) \times Sp(1))$.*

(ii) *The Stiefel manifold $\widetilde{V}_4(\mathbb{R}^{n+1})$ admits an (integrable) complex structure, projecting to the complex structure of $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$.*

Here $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$ is the twistor space of the quaternion Kähler Wolf space $\widetilde{Gr}_4(\mathbb{R}^{n+1})$ given by the $Sp(1)$ reduction on $\mathbb{H}P^n$. $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$ is known to be a complex submanifold of the Kähler-Einstein Grassmannian $Gr_2(\mathbb{C}^{n+1})$, see for example [25] p. 14 or [23] p. 702.

Statements 3.1, 3.2 give, in particular, some fibrations that can be collected into a diagram as follows. Here $V_k(\mathbb{C}^{n+1})$ and $\widetilde{V}_k(\mathbb{R}^{n+1})$ denote the Stiefel manifolds of k -frames in \mathbb{C}^{n+1} and of oriented k -frames in \mathbb{R}^{n+1} , respectively.

Corollary 3.1. *There is a commutative diagram*

$$\begin{array}{ccccc} \widetilde{V}_4(\mathbb{R}^{n+1}) & \hookrightarrow & V_2(\mathbb{C}^{n+1}) & \hookrightarrow & S^{4n+3} \\ \downarrow S^3 & & \downarrow S^3 & & \downarrow S^3 \\ \nu^{-1}(0) & \hookrightarrow & \mu^{-1}(0) & \hookrightarrow & \mathbb{H}P^n & S^{2N+1} \\ \downarrow S^1 & & \downarrow S^1 & & & \swarrow \\ Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})} & \hookrightarrow & Gr_2(\mathbb{C}^{n+1}) & \hookrightarrow & \mathbb{C}P^N \\ \downarrow S^2 & & & & \\ \widetilde{Gr}_4(\mathbb{R}^{n+1}) & & & & \end{array}$$

of principal S^1 and S^3 -bundles and Riemannian submersions. The zero level sets $\mu^{-1}(0)$ and $\nu^{-1}(0)$ are thus total spaces of induced Hopf S^1 -bundles and, as such, are minimal submanifolds of the sphere S^{2N+1} .

4. THE PROOFS

Proof of Theorem 3.1 (i) and (ii). The first observation is:

$Foc_{\mathbb{H}P^1}\mathbb{C}P^1$ is the level set of the moment map associated to the standard $U(1)$ -action on $\mathbb{H}P^1$.

In the homogeneous coordinates $[h_0 : h_1]$ of $\mathbb{H}P^1$, using the notations from formula (2.1), $\mathbb{C}P^1$ is given by:

$$(4.1) \quad w_0 = 0, w_1 = 0 \text{ or } \gamma_0 = \delta_0 = \gamma_1 = \delta_1 = 0.$$

The pair $\mathbb{C}P^1 \subset \mathbb{H}P^1$ can be identified with $S^2 \subset S^4$, and the focal set of a totally geodesic sphere S^p in S^n is the unit S^{n-p-1} in the $(n-p)$ -dimensional orthogonal complement of the \mathbb{R}^{p+1} containing S^p ([33], p. 286). This identification can be made explicit by using the coordinate $h = \alpha + \beta i + \gamma j + \delta k = h_0 h_1^{-1}$ in the affine line $h_1 \neq 0$, whose corresponding real coordinates are:

$$\begin{aligned} \alpha &= \frac{\alpha_0 \alpha_1 + \beta_0 \beta_1 + \gamma_0 \gamma_1 + \delta_0 \delta_1}{r^2}, \quad \beta = \frac{\alpha_1 \beta_0 - \alpha_0 \beta_1 - \gamma_0 \delta_1 + \gamma_1 \delta_0}{r^2}, \\ \gamma &= \frac{\alpha_1 \gamma_0 - \alpha_0 \gamma_1 - \delta_0 \beta_1 + \beta_0 \delta_1}{r^2}, \quad \delta = \frac{\alpha_1 \delta_0 - \alpha_0 \delta_1 - \gamma_1 \beta_0 + \gamma_0 \delta_1}{r^2}, \end{aligned}$$

where $r^2 = \alpha_1^2 + \beta_1^2 + \gamma_1^2 + \delta_1^2$. If (x_1, \dots, x_5) are the standard coordinates on \mathbb{R}^5 , the (inverse) stereographic projection $\mathbb{R}^4 \rightarrow S^4$ reads:

$$(4.2) \quad (\alpha, \beta, \gamma, \delta) \mapsto (x_1, \dots, x_5) = \left(\frac{2\alpha}{r^2 + 1}, \frac{2\beta}{r^2 + 1}, \frac{2\gamma}{r^2 + 1}, \frac{2\delta}{r^2 + 1}, \frac{r^2 - 1}{r^2 + 1} \right).$$

Thus, if $x_3 = x_4 = 0$ are the equations of a totally geodesic S^2 in S^4 , the focal set of S^2 in S^4 is the S^1 given by the equations $x_1 = x_2 = x_5 = 0$, corresponding to

$$(4.3) \quad \alpha = \beta = 0, \quad \gamma^2 + \delta^2 = 1.$$

On the other hand, the moment map μ on $\mathbb{H}P^1$ can be written in the non-homogeneous coordinate $h = h_0 h_1^{-1}$ as $\mu = \bar{h} i h + i$. Its level set $\mu^{-1}(0)$ is thus:

$$(4.4) \quad \alpha_1^2 + \beta_1^2 - \gamma_1^2 - \delta_1^2 + 1 = 0, \quad \alpha\delta - \beta\gamma = 0, \quad \alpha\gamma + \beta\delta = 0,$$

and the systems (4.3) and (4.4) are equivalent.

We now treat the case $n > 1$. From the definition of the focal set, we only need to look at geodesics normal to $\mathbb{C}P^n$. These are normal to all the complex projective lines $\mathbb{C}L^1 \subset \mathbb{C}P^n$, thus their focal points with respect to $\mathbb{C}P^n$ are also focal points for all the lines $\mathbb{C}L^1$ contained in $\mathbb{C}P^n$. Any such line $\mathbb{C}L^1$ in $\mathbb{H}P^n$ belongs to a unique quaternionic projective line $\mathbb{H}L^1$ and the latter is totally geodesic in $\mathbb{H}P^n$. It follows that any geodesic that is normal to a $\mathbb{C}L^1$ and tangent to the corresponding $\mathbb{H}L^1$ at a given point, remains tangent to $\mathbb{H}L^1$ for its entire length. Hence, if we show that through any point x of $\mathbb{C}P^n$ and for any $v \in T_x^\perp \mathbb{C}P^n$ there exists a projective quaternionic line $\mathbb{H}L^1$ containing v , we may deduce:

$Foc_{\mathbb{H}P^n} \mathbb{C}P^n$ is the union of all the focal sets $Foc_{\mathbb{H}L^1} \mathbb{C}L^1$.

This can be seen from the diagram

$$\begin{array}{ccc} \mathbb{C}^{n+1} \setminus \{0\} & \hookrightarrow & \mathbb{H}^{n+1} \setminus \{0\} \\ \downarrow & & \downarrow \\ \mathbb{C}P^n & \hookrightarrow & \mathbb{H}P^n. \end{array}$$

by looking at the complex planes $L_2^{\mathbb{C}}$ in \mathbb{C}^{n+1} containing the fibre \mathbb{C}^* and the corresponding hypercomplex 2-planes $L_2^{\mathbb{H}}$ in \mathbb{H}^{n+1} containing the fibre \mathbb{H}^* . Observe that, for any vector $\vec{v} \in \mathbb{H}^{n+1}$ which is normal to the standard embedded \mathbb{C}^{n+1} , there exist a $L_2^{\mathbb{C}}$ and a $L_2^{\mathbb{H}}$ with $\text{span}_{\mathbb{R}}\{L_2^{\mathbb{C}}, \vec{v}\} \subset L_2^{\mathbb{H}}$: indeed, if $\mathbb{H}^{n+1} = \text{span}_{\mathbb{H}}\{\vec{e}_0, \dots, \vec{e}_n\}$ and $\mathbb{C}^{n+1} = \text{span}_{\mathbb{C}}\{\vec{e}_0, \dots, \vec{e}_n\}$, we have:

$$\vec{v} \perp \mathbb{C}^{n+1} \text{ if and only if } \vec{v} = \sum_{a=0}^n \lambda_a j \vec{e}_a + \mu_a k \vec{e}_a.$$

Hence, if

$$\vec{w} = -j \vec{v} = \sum_{a=0}^n \lambda_a \vec{e}_a - \mu_a i \vec{e}_a,$$

we obtain

$$L_2^{\mathbb{C}} = \text{span}_{\mathbb{C}}\{\vec{e}_0, \vec{w}\}$$

and

$$L_2^{\mathbb{H}} = \text{span}_{\mathbb{R}}\{\vec{e}_0, i \vec{e}_0, j \vec{e}_0, k \vec{e}_0, \vec{w}, i \vec{w}, j \vec{w}, k \vec{w}\}$$

satisfy the condition.

The identification of $Foc_{\mathbb{H}P^n} \mathbb{C}P^n$ with $\mu^{-1}(0)$ is then completed by the following observation:

The subset $\mu^{-1}(0) \subset \mathbb{H}P^n$ is the union of the zero sets of the moment maps μ_1 associated to the standard $U(1)$ -action on all the projective quaternionic lines $\mathbb{H}L^1 \subset \mathbb{H}P^n$.

Let $\{p_0, p_1, \dots, p_n\}$ be the canonical frame of $\mathbb{H}P^n$ with unit point u and let $[h_0 : \dots : h_n]$ be the homogeneous coordinates with respect to this frame. Accordingly, the moment map reads $\mu(h) = \sum_{a=0}^n \bar{h}_a i h_a$. Fix a $q \in \mu^{-1}(0)$, i.e. $\sum_{a=0}^n \bar{q}_a i q_a = 0$, and note that q cannot be real; however, we may suppose $q_0 \neq 0$.

Let $\mathbb{H}L^1 \cong \mathbb{H}P^1$ be the quaternionic projective line through q and p_1 . To compare the intersection $\mu^{-1}(0) \cap \mathbb{H}L^1$ with the zero level set of the moment map in $\mathbb{H}L^1 \cong \mathbb{H}P^1$, we change the frame in $\mathbb{H}P^n$ as follows. We want new homogeneous coordinates $[k_0 : k_1 : \dots : k_n]$ such that $q = [1 : j : 0 : \dots : 0]$ and the coordinates of p_1, \dots, p_n remain unchanged. With the new unit point $v = [q_0 : 1 : -1 : -1 : \dots : -1]$, the matrix of the change of coordinates is:

$$A = \begin{pmatrix} q_0 & 0 & 0 & \dots & 0 \\ q_1 - j & 1 & 0 & \dots & 0 \\ q_2 & 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ q_n & 0 & 0 & \dots & -1 \end{pmatrix}.$$

Thus, ${}^t[h_0 : \dots : h_n] = A^t[k_0 : \dots : k_n]$, and the moment map is:

$$\mu = \sum_{a=0}^n \bar{h}_a i h_a = \bar{k}_0 \left(\sum_{a=0}^n \bar{q}_a i q_a \right) k_0 - \bar{k}_0 j i j k_0 - \sum_{b=1}^n \bar{k}_b i k_b = - \sum_{b=0}^n \bar{k}_b i k_b.$$

It follows that $\mu^{-1}(0)$ is invariant under these projective changes of coordinates. As the coordinates on $\mathbb{H}L^1$ are $[k_0 : k_1]$, its moment map is $\mu_1 = -\bar{k}_0 i k_0 - \bar{k}_1 i k_1$ with the same level set $\mu_1^{-1}(0)$ described for $\mathbb{H}P^1$. This establishes the inclusion $\mu^{-1}(0)|_{\mathbb{H}L^1} \subset \mu_1^{-1}(0)$. As the converse inclusion is clear, for any projective line $\mathbb{H}L^1$ in $\mathbb{H}P^n$, the proof of the identification $Foc_{\mathbb{H}P^n} \mathbb{C}P^n = \mu^{-1}(0)$ is complete.

Next, we prove that $Foc_{\mathbb{H}P^n} \mathbb{C}P^n$ is isometric to the total space of the induced Hopf bundle over the Grassmannian $Gr_2(\mathbb{C}^{n+1})$. In fact, since $Foc_{\mathbb{H}P^n} \mathbb{C}P^n$ is simply connected ([7], p. 17), the existence of a diffeomorphism with the induced Hopf bundle is a consequence of the following observation:

Let $\pi : P \rightarrow Gr_2(\mathbb{C}^{n+1})$ be a principal circle bundle with simply connected P . Then P is diffeomorphic to the total space of the induced

Hopf bundle of $S^{2N+1} \rightarrow \mathbb{C}P^N$, $N = \binom{n+1}{2} - 1$, via the Plücker embedding $Gr_2(\mathbb{C}^{n+1}) \hookrightarrow \mathbb{C}P^N$.

In fact, the principal S^1 -bundles over the base B are classified by their Chern class in $H^2(B, \mathbb{Z})$. Since $H^2(Gr_2(\mathbb{C}^{n+1}), \mathbb{Z}) \cong \mathbb{Z}$ is generated by the class of the Kähler form, one can denote by P_a the S^1 -bundle over $Gr_2(\mathbb{C}^{n+1})$ associated to $a \in \mathbb{Z}$. Observe that P_1 (resp. P_{-1}) is the circle bundle associated to the canonical line bundle $O_{Gr_2(\mathbb{C}^{n+1})}(1)$ (resp. its dual $O_{Gr_2(\mathbb{C}^{n+1})}(-1)$). But dual complex line bundles are diffeomorphic as real vector bundles. Hence P_1 is diffeomorphic to P_{-1} .

Let us now show that if P_a is simply connected, then $a = \pm 1$. From $\pi_1(P) = 0$, we have $H^1(P, \mathbb{Z}) = 0$ and $H^2(P, \mathbb{Z})$ torsion free. Thus the Gysin sequence of the S^1 -bundle π :

$$\begin{aligned} 0 \rightarrow H^1(P, \mathbb{Z}) \rightarrow H^0(Gr_2(\mathbb{C}^{n+1}), \mathbb{Z}) \rightarrow H^2(Gr_2(\mathbb{C}^{n+1}), \mathbb{Z}) \\ \rightarrow H^2(P, \mathbb{Z}) \rightarrow 0 \end{aligned}$$

reduces to:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cup c_1} \mathbb{Z} \longrightarrow H^2(P, \mathbb{Z}) \longrightarrow 0,$$

where c_1 is the Chern class of π . Hence, $c_1 \neq \pm 1$ implies $H^2(P, \mathbb{Z}) \cong \mathbb{Z}_n$ for some $n \geq 2$. Consequently, c_1 is, up to sign, the Chern class of the induced Hopf bundle.

We now look at the metric g_1 inherited from (S^{2N+1}, can) by the total space $Foc_{\mathbb{H}P^n}(\mathbb{C}P^n)$ of the induced Hopf bundle and at the metric g induced on $Foc_{\mathbb{H}P^n}(\mathbb{C}P^n)$ from $\mathbb{H}P^n$. Note that both $(Foc_{\mathbb{H}P^n}(\mathbb{C}P^n), g_1)$ and $(Foc_{\mathbb{H}P^n}(\mathbb{C}P^n), g)$ are Riemannian submersions with geodesic fibres S^1 over $Gr_2(\mathbb{C}^{n+1})$ (cf. [31] for g_1 and [7] for g). Thus Theorem 9.59 in [6], p. 249, can be used to conclude that $g = g_1$. This completes the proof of (i) and (ii) in Theorem 3.1.

Remark 4.1. An alternative way of recognizing that $Foc_{\mathbb{H}P^n}\mathbb{C}P^n = \mu^{-1}(0)$ is to look at the standard isometric action of $SU(n+1)$ on $\mathbb{H}P^n$ and to see that the two subsets are obtained as homogeneous spaces of $SU(n+1)$ with the same isotropy groups. The homogeneous space is in fact $\frac{SU(n+1)}{SU(2) \times SU(n-1)}$, which was shown in [7], p. 17, to be a singular orbit of the action of $SU(n+1)$ (it was denoted there by Q^n). A similar description of $\mu^{-1}(0)$ as a homogeneous space is given in [1], p. 65, in relation with the problem of studying local compatible complex structures in $\mathbb{H}P^n$. Compare also with [36], p. 171, where $\frac{SU(n+1)}{SU(2) \times SU(n-1)}$ is called the "Grassmannian of *oriented* two-planes" of \mathbb{C}^{n+1} .

Remark 4.2. We proved in [31] that the metric g_1 is Sasakian and η -Einstein, and this is in accordance with formulas following Proposition 9 in [7]. As for the extrinsic geometry of $Foc_{\mathbb{H}P^n}(\mathbb{C}P^n)$, J. Berndt proves (Corollary 1 in [7]) that the immersion in $\mathbb{H}P^n$ is minimal. It is interesting to observe that the same holds for the immersion $(Foc_{\mathbb{H}P^n}\mathbb{C}P^n, g_1)$ in (S^{2N+1}, can) , see [31].

Proof of Theorem 3.2 (i). The identification of $\nu^{-1}(0)$ with the intersection M is an immediate consequence of the definition of ν and of the first statement of Theorem 3.1. To see that $\nu^{-1}(0)$ is isometric to the induced Hopf bundle discussed above, note that the last observation in the proof of Theorem 3.1 (i) and (ii) still holds for principal circle bundles with simply connected P over any complex algebraic projective submanifold B of $\mathbb{C}P^N$ with $H^2(B, \mathbb{Z}) \cong \mathbb{Z}$. This applies in particular to $B = Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$, as soon as one recognizes that $\nu^{-1}(0)$ is simply connected. To see this, observe that $\nu^{-1}(0)$ can be regarded as the homogeneous space $\frac{SO(n+1)}{SO(n-3) \times Sp(1)}$ via the transitive action of $SO(n+1)$ on $\nu^{-1}(0)$. This last action comes, in fact, from the natural action of $SO(n+1)$ on $\mathbb{H}P^n$ and J. Berndt's observation that $SU(n+1)$ acts transitively on $Foc_{\mathbb{H}P^n}(\mathbb{C}P^n)$. His argument can be triplicated to produce transitive actions of groups $SU_i(n+1)$, $SU_j(n+1)$, $SU_k(n+1)$ (the first one is the standard $SU(n+1)$ in $Sp(n+1)$, the other two are similarly defined by interchanging the rôles of the unit quaternions i, j, k) on the zero level sets $\mu_i^{-1}(0)$, $\mu_j^{-1}(0)$, $\mu_k^{-1}(0)$. The isotropy subgroup of the action on $\nu^{-1}(0)$ at the point $[1 : i : j : k : 0 : \dots : 0]$ is then $Sp(1) \times SO(n-3)$. The homotopy sequence associated to the homogeneous manifold $\frac{SO(n+1)}{Sp(1) \times SO(n-3)}$ thus shows that $\nu^{-1}(0)$ is simply connected, so that it is diffeomorphic with the total space of the induced Hopf bundle, now over the Fano manifold $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$.

As an induced S^1 -Hopf bundle, $\nu^{-1}(0)$ has a Sasakian η -Einstein structure. This can be deformed to a Sasakian-Einstein metric, which is still a Riemannian submersion after rescaling the standard Kähler-Einstein metric of $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$ [31]. On the other hand, the composition of the fiberings

$$\nu^{-1}(0) \xrightarrow{S^1} Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})} \xrightarrow{S^2} \widetilde{Gr}_4(\mathbb{R}^{n+1})$$

is an $SO(3)$ -bundle which endows $\nu^{-1}(0)$ with a 3-Sasakian structure via the inversion Theorem 4.6 of [10].

Proof of Theorem 3.1 (iii) and Theorem 3.2 (ii). In both cases we construct a complex structure on the total space of an S^3 bundle over a Sasakian manifold, induced by the Hopf bundle $S^{4n+3} \rightarrow \mathbb{H}P^n$.

Both Stiefel manifolds under discussion are induced Hopf S^3 -bundles, as is easily recognized by regarding them as homogeneous spaces. Thus the induced homogeneous metrics make them Riemannian submersions with fibers S^3 . More generally:

Proposition 4.1. *Let $\pi : P \rightarrow B$ be a principal S^3 -bundle induced by the Hopf bundle $S^{4n+3} \rightarrow \mathbb{H}P^n$, and let g^B, ∇^B be the induced metric and Levi Civita connection on $B \subset \mathbb{H}P^n$. Assume that B admits a Killing vector field ξ such that $\varphi = \nabla^B \xi$ defines on B a Sasakian structure. Then P admits an almost complex structure.*

Proof. Let g^P be the induced metric on $P \subset S^{4n+3}$, so that π is a Riemannian submersion. For any $X \in \mathcal{X}(B)$ we let X^* be its horizontal lift on P . Let ξ_1, ξ_2, ξ_3 be the unit Killing vector fields which give the usual 3-Sasakian structure of the fibers S^3 and η_1, η_2, η_3 their duals with respect to the canonical metric of S^3 . The ξ_i may be viewed as vector fields on P . Let $\hat{\eta}_i$ be their dual forms with respect to the metric g^P . Restricted to any fibre, the $\hat{\eta}_i$ coincide with the η_i . The usual splitting $TP = \mathcal{V} \oplus \mathcal{H}$ into vertical and horizontal parts may be refined to:

$$TP = \text{span}\{\xi_1, \xi_2, \xi_3\} \oplus \text{span}\{\xi^*\} \oplus \mathcal{H}',$$

where \mathcal{H}' represents the horizontal vector fields orthogonal to the horizontal lift ξ^* of ξ .

Define the almost complex structure J on P by:

- $J\xi_1 = \xi_2, \quad J\xi_2 = -\xi_1,$
- $J\xi_3 = \xi^*, \quad J\xi^* = -\xi_3,$
- $JX^* = (\varphi X)^*$ for any $X \in \mathcal{X}(B)$ orthogonal to ξ .

Note that for $X \perp \xi$, X^* is a section of \mathcal{H}' . As the restriction of φ to ξ^\perp is an endomorphism of ξ^\perp , the last item in the definition is consistent. One easily shows that $J^2 = -1$ and J is compatible with g . \square

To discuss the integrability of the constructed J , we follow the discussion developed [15] for an almost hypercomplex structure, computing the Nijenhuis tensor field on all possible combinations of vertical and/or horizontal vector fields.

Note first that the horizontal distribution \mathcal{H} is an $sp(1)$ -connection in the induced Hopf S^3 bundle $P \rightarrow B$. This follows from the fact that the bracket of any horizontal X^* with a vertical vector field is horizontal, a consequence of the Killing property of the ξ_i with respect to g^P . Thus, in particular we get that for $i = 1, 2, 3$:

$$(4.5) \quad \hat{\eta}_k[\xi_i, X^*] = \hat{\eta}_k[\xi_i, \xi^*] = 0,$$

for $k = 1, 2, 3$. Now we can prove:

Proposition 4.2. *Assume that the curvature form Ω of the $sp(1)$ -connection \mathcal{H} satisfies the following conditions:*

1) $\Omega((\varphi X)^*, (\varphi Y)^*) = \Omega(X^*, Y^*)$, i.e. Ω is of type $(1, 1)$ with respect to J ,

2) $\Omega(X^*, \xi^*) = 0$ for any $X \perp \xi$.

Then the almost complex structure J is integrable.

Remark 4.3. Conditions 1), 2) in the former Proposition express a compatibility between the Sasakian structure of the base and the bundle structure of P . Since the vertical components of Ω are precisely the $d\hat{\eta}_i$, the two conditions give corresponding equations for $d\hat{\eta}_i$. Moreover, the condition $d\hat{\eta}_i((\varphi X)^*, (\varphi Y)^*) = d\hat{\eta}_i(X^*, Y^*)$ is easily checked to be equivalent with $d\hat{\eta}_i((\varphi X)^*, Y^*) + d\hat{\eta}_i(X^*, (\varphi Y)^*) = 0$.

Now we can give the proof of the Proposition 4.2:

Proof. We compute the Nijenhuis tensor field of J :

$$[J, J](A_1, A_2) = [A_1, A_2] + J[JA_1, A_2] + J[A_1, JA_2] - [JA_1, JA_2]$$

for all possible pairs (A_1, A_2) , noting that, due to the tensorial character of $[J, J]$, when dealing with horizontal (resp. vertical) vector fields it is enough to work with basic ones (resp. with ξ_i , $i = 1, 2, 3$).

1. Let first $A_1 = X^*$, $A_2 = Y^*$, $X, Y \perp \xi$. Denoting by $\hat{\eta}$ the dual of ξ^* we get:

$$(4.6) \quad [X^*, Y^*] = [X^*, Y^*]' + \hat{\eta}([X^*, Y^*])\xi^* + \text{vertical part.}$$

where the $'$ denotes the \mathcal{H}' part. By π -corelation, $[X^*, Y^*] = [X, Y]^*$. Moreover, from $\hat{\eta}(X^*) = \hat{\eta}(Y^*) = 0$ we get $\hat{\eta}([X^*, Y^*]) = -d\hat{\eta}(X^*, Y^*)$. The vertical part of $[X^*, Y^*]$ must be of the form $\sum_{i=1}^3 \hat{\eta}_i([X^*, Y^*])\xi_i$. Hence:

$$(4.7) \quad [X^*, Y^*] = [X^*, Y^*]' - d\hat{\eta}(X^*, Y^*)\xi^* - \sum_{i=1}^3 d\hat{\eta}(X^*, Y^*)\xi_i.$$

By similar computations:

$$\begin{aligned}
J[JX^*, Y^*] &= (\varphi[\varphi X, Y])^{*'} + d\hat{\eta}((\varphi X)^*, Y^*)\xi_3 - \\
&\quad - d\hat{\eta}_1((\varphi X)^*, Y^*)\xi_2 + d\hat{\eta}_2((\varphi X)^*, Y^*)\xi_1 - d\hat{\eta}_3((\varphi X)^*, Y^*)\xi^*, \\
J[X^*, JY^*] &= (\varphi[X, \varphi Y])^{*'} + d\hat{\eta}(X^*, (\varphi Y)^*)\xi_3 - \\
&\quad - d\hat{\eta}_1(X^*, (\varphi Y)^*)\xi_2 + d\hat{\eta}_2(X^*, (\varphi Y)^*)\xi_1 - d\hat{\eta}_3(X^*, (\varphi Y)^*)\xi^*, \\
[JX^*, JY^*] &= [\varphi X, \varphi Y]^{*'} - d\hat{\eta}((\varphi X)^*, (\varphi Y)^*)\xi^* \\
&\quad - \sum_{i=1}^3 d\hat{\eta}_i((\varphi X)^*, (\varphi Y)^*)\xi_i.
\end{aligned}$$

Hence, using the $(1, 1)$ character of $d\hat{\eta}_i$ and Remark 4.3, we find

$$[J, J](X^*, Y^*) = [\varphi X, \varphi Y]^{*'} - \{d\hat{\eta}(X^*, Y^*) - d\hat{\eta}((\varphi X)^*, (\varphi Y)^*)\}\xi^*.$$

As we know $[\varphi X, \varphi Y] + 2d\eta(X, Y)\xi = 0$ (this is the normality condition of the Sasakian structure of B) the horizontal lift of this (null) tensor field is zero, hence also its component in \mathcal{H}' is zero. But this is precisely $[\varphi X, \varphi Y]^{*'}$.

On the other hand, on any Sasakian manifold one has:

$$d\eta(X, Y) = g(X, \varphi Y), \quad \varphi^2 X = -X + \eta(X)\xi,$$

thus $d\eta(X, \varphi Y) + d\eta(\varphi X, Y) = 0$ and $d\eta(\varphi X, \varphi Y) - d\eta(X, Y) = 0$. By horizontally lifting these equations we obtain the annulation of the ξ^* component, hence $[J, J](X^*, Y^*) = 0$.

2. Let now $A_1 = X^*$, $A_2 = \xi^*$ ($X \perp \xi$). Then:

$$\begin{aligned}
[J, J](X^*, \xi^*) &= [X^*, \xi^*] + J[JX^*, \xi^*] + J[X^*, J\xi^*] - [JX^*, J\xi^*] = \\
&= [X^*, \xi^*] + J[(\varphi X)^*, \xi^*] - J[X^*, \xi_3] + [(\varphi X)^*, \xi_3] = \\
&= [X^*, \xi^*] + J[(\varphi X)^*, \xi^*],
\end{aligned}$$

as the last two brackets are zero by (4.5). As above, using Remark 4.3 and $d\hat{\eta}_i(X^*, \xi^*) = 0$ (condition 2) in the statement) we obtain:

$$[J, J](X^*, \xi^*) = ([X, \xi] + \varphi[\varphi X, \xi])^{*'} = 0,$$

because, as on a Sasakian manifold $\varphi\xi = 0$, we can add in the paranthesis the terms $[X, \varphi\xi] - [\varphi X, \varphi\xi]$ obtaining $([X, \xi] + \varphi[\varphi X, \xi] + \varphi[X, \varphi\xi] - [\varphi X, \varphi\xi])^{*'} = ([\varphi, \varphi](X, \xi))^{*'} = 0$ by the normality condition on B .

3. We now choose $A_1 = X^*$ and $A_2 = \xi_i$ ($i = 1, 2$). We have:

$$[J, J](X^*, \xi_1) = [X^*, \xi_1] + J[JX^*, \xi_1] + J[X^*, J\xi_1] - [JX^*, J\xi_1] = 0,$$

because \mathcal{H} is a $sp(1)$ -connection. Similarly for $[J, J](X^*, \xi_2) = 0$.

4. For $A_1 = X^*$ and $A_2 = \xi_3$ we find:

$$\begin{aligned} [J, J](X^*, \xi_3) &= [X^*, \xi_3] + J[JX^*, \xi_3] + J[X^*, J\xi_3] - [JX^*, J\xi_3] = \\ &= J[X^*, \xi^*] - [(\varphi X)^*, \xi^*], \end{aligned}$$

the brackets with ξ_3 being zero by (4.5). The horizontal component of the remaining two brackets is $([\varphi[X, \xi] - [\varphi X, \xi]]^* - d\hat{\eta}_3(X^*, \xi^*)\xi_i$. The ξ^* component as well as the vertical component vanish by assumption 2) in the statement. Finally, using Sasakian identities and the normality condition on B we have $([\varphi[X, \xi] - [\varphi X, \xi]]^*)' = 0$.

5. Immediate computation shows that in the remaining "mixed" case $[J, J](\xi_i, \xi^*) = 0$.

6. We are left with the computation of $[J, J]$ on vertical fields. Obviously $[J, J](\xi_1, \xi_2) = 0$. Then

$$\begin{aligned} [J, J](\xi_1, \xi_3) &= [\xi_1, \xi_3] + J[J\xi_1, \xi_3] + J[\xi_1, J\xi_3] - [J\xi_1, J\xi_3] \\ &= [\xi_1, \xi_3] + J[\xi_2, \xi_3] + J[\xi_1, \xi^*] - [\xi_2, \xi^*] = 0 \end{aligned}$$

by $[\xi, \xi_j] = 2\varepsilon_{ijk}\xi_k$ and (4.5). Same arguments show that $[J, J](\xi_2, \xi_3) = 0$, thus completing the proof. \square

Remark 4.4. The Kähler form of (P, g, J) is

$$\omega = d\pi^*\eta + \pi^*\eta \wedge \eta_3 - d\eta_3;$$

this shows that $d\omega \neq 0$, hence the structure is not Kählerian. A similar computation proves that $L_{\xi^*}J = L_{\xi_3}J = 0$, thus ξ^* and ξ_3 are infinitesimal automorphisms of the constructed complex structure.

Remark 4.5. The complex structure J on P depends on the choice of a 3-Sasakian structure on the fibre. But one can see that different choices of 3-Sasakian triples $\{\xi_1, \xi_2, \xi_3\}$ produce complex structures that are conjugated in $End(TP)$.

We can now go back to the Stiefel manifolds $V_2(\mathbb{C}^{n+1})$ and $\tilde{V}_4(\mathbb{R}^{n+1})$, and complete the proof of Theorems 3.1 (iii) and 3.2 (ii). We have just to verify that the induced Hopf bundles $V_2(\mathbb{C}^{n+1}) \rightarrow \mu^{-1}(0)$ and $\tilde{V}_4(\mathbb{R}^{n+1}) \rightarrow \nu^{-1}(0)$ inherit from the inclusions $\nu^{-1}(0) \subset \mu^{-1}(0) \subset \mathbb{H}P^n$ horizontal distributions \mathcal{H} satisfying the curvature properties of Proposition 4.2. Now property 1) simply express that the $sp(1)$ -connection given by \mathcal{H} is part of a $u(2)$ -connection in the bundles $V_2(\mathbb{C}^{n+1}) \rightarrow Gr_2(\mathbb{C}^{n+1})$ and $\tilde{V}_4(\mathbb{R}^{n+1}) \rightarrow Z_{\tilde{Gr}_4(\mathbb{R}^{n+1})}$, a fact easily recognized as in the case of hypercomplex structures in $V_2(\mathbb{C}^{n+1})$ (cf.[14], proof of Thm

1.10, as well as [15], Thm. 1.13). The meaning of property 2) is that the curvature of such a $sp(1)$ -connection is given by a $(1,1)$ -form with respect to the almost complex structure J . This follows for example from the computation carried out in [34], p. 63, for the Hopf bundle. One has to take into account that the rôle of the $U(1)$ and of the $Sp(1)$ -actions on both Stiefel manifolds correspond to the standard basic choice of left and right multiplication by scalars on \mathbb{H}^{n+1} . Then both on $V_2(\mathbb{C}^{n+1})$ and on $\tilde{V}_4(\mathbb{R}^{n+1})$ the almost complex structure J satisfies the compatibility conditions with the Sasakian structures of $\mu^{-1}(0)$ and $\nu^{-1}(0)$, as expressed by conditions 1) and 2) of Proposition 4.2. Note that the complex structure obtained in this way on $V_2(\mathbb{C}^{n+1})$ projects to the complex structure underlying the Kähler structure of $Gr_2(\mathbb{C}^{n+1})$. This latter is well known to be not compatible with the quaternion Kähler structure of this Grassmannian. But it is precisely this quaternion Kähler structure which is lifted to a 3-Sasakian structure and then, by means of an appropriate circle bundle, to a hypercomplex structure on $V_2(\mathbb{C}^{n+1})$, cf. [2], [14]. Hence our complex structure is not compatible with the standard hypercomplex one of $V_2(\mathbb{C}^{n+1})$. This completes the proof of Theorem 3.1 (iii) and 3.2 (ii).

Proof of Corollary 3.1. It remains only to show that the total space of an induced S^1 Hopf bundle is minimal in (S^{2N-1}, can) . In fact, more generally, in a commutative diagram

$$\begin{array}{ccc} \overline{N} & \xrightarrow{\bar{i}} & \overline{M} \\ \pi_N \downarrow & & \downarrow \pi_M \\ N & \xrightarrow{i} & M \end{array}$$

of immersions i, \bar{i} and Riemannian submersions π_N, π_M with totally geodesic fibres, we see that N is minimal in M if and only if \overline{N} is minimal in \overline{M} . This follows by a direct computation of the mean curvature vector fields of i and \bar{i} using the Gauss formula of a submanifold and formula (9.25 a) in [6].

5. FURTHER OBSERVATIONS

As mentioned in the introduction, the zero level set $\mu^{-1}(0) \subset \mathbb{H}P^2$ is diffeomorphic to a sphere S^5 and the projection to the reduced manifold can be identified with the Hopf fibration $S^5 \rightarrow \mathbb{C}P^2$. Going to the next case $n = 3$, the projection $\mu^{-1}(0) \rightarrow Gr_2(\mathbb{C}^4)$, now an induced Hopf fibration, can be described by looking at the families of submanifolds in $Gr_2(\mathbb{C}^4)$ that are either Kähler-Einstein or quaternion

Kähler. The Grassmannian $Gr_2(\mathbb{C}^4)$ admits in fact a natural complex Kähler structure, as well as two distinct quaternion Kähler structures induced via the isomorphism of vector bundles $TGr_2(\mathbb{C}^4) \cong V \otimes V^\perp$ from the (almost) hypercomplex structure on the tautological vector bundle V or on its orthogonal complement V^\perp (cf. [27]). The families of submanifolds we want to look at on $Gr_2(\mathbb{C}^4)$ are described as follows (see [7], [8], [26], [35]).

There are two families \mathcal{F} , \mathcal{F}' of complex projective planes $\mathbb{C}P^2$, $\mathbb{C}P^{2'}$, a family \mathcal{F}'' of products $\mathbb{C}P^1 \times \mathbb{C}P^1$, and a family \mathcal{F}''' of spheres S^4 ; they are given by:

$$\begin{aligned}\mathcal{F} &= \{\text{planes contained in a 3-space of } \mathbb{C}^4\}, \\ \mathcal{F}' &= \{\text{planes through a line of } \mathbb{C}^4\}, \\ \mathcal{F}'' &= \{\text{planes that are invariant for a hypercomplex} \\ &\quad \text{structure } J \text{ of } \mathbb{C}^4\}, \\ \mathcal{F}''' &= \{\text{planes given by pairs of lines in two fixed orthogonal} \\ &\quad \text{planes of } \mathbb{C}^4\}.\end{aligned}$$

All these 4-dimensional submanifolds of $Gr_2(\mathbb{C}^4)$ have nice intersection properties, some of which can also be formulated in terms of projective geometry of the lines in the 3-dimensional space $\mathbb{C}P^3$, the context where the Klein quadric $\mathbb{C}Q^4 \subset \mathbb{C}P^5$, isometric to $Gr_2(\mathbb{C}^4)$, was first introduced. Instead of listing these intersection properties on $Gr_2(\mathbb{C}^4)$ (cf. [26], pp. 508-512, for some of them; the remaining ones can be similarly deduced), we formulate the corresponding properties for the families of 5-dimensional submanifolds obtained at the Sasakian Einstein level as induced Hopf bundles over the members of families $\mathcal{F}, \dots, \mathcal{F}'''$.

Proposition 5.1. *The 9-dimensional Sasakian Einstein focal set $Foc_{\mathbb{H}P^3} \mathbb{C}P^3 \cong \mu^{-1}(0) \subset \mathbb{H}P^3$ contains the following families of 5-dimensional submanifolds, each of which fibers in circles over a complex or a quaternionic submanifold of $Gr_2(\mathbb{C}^4)$. There are two families \mathcal{E} , \mathcal{E}' of Sasakian 5-spheres S^5 , S'^5 , a family \mathcal{E}'' of Sasakian products $S^3 \times S^2$ and a family \mathcal{E}''' of products $S^4 \times S^1$. The induced metrics on members of the families \mathcal{E} , \mathcal{E}' , \mathcal{E}'' are Sasakian η -Einstein, and can be modified to Sasakian Einstein metrics by formula (2.1). The intersection properties of these 5-dimensional submanifolds are the following.*

(a) *Pairs of 5-spheres in the same family intersect in a circle, and pairs of 5-spheres in different families either do not intersect or intersect in an S^3 .*

(b) Pairs of submanifolds of type $S^3 \times S^2$ intersect either in an S^3 or in a pair of disjoint circles. A submanifold of type $S^3 \times S^2$ intersects a 5-sphere in an S^3 .

(c) Any submanifold of type $S^4 \times S^1$ intersects a 5-sphere in a circle, intersects an $S^3 \times S^2$ in two disjoint circles, and any pair of submanifolds of type $S^4 \times S^1$ intersect in two disjoint circles.

This kind of geometry of submanifolds, now described for the level set $\mu^{-1}(0) \subset \mathbb{H}P^3$, can be formulated for all the zero level sets of moment maps appearing in the diagram of Corollary 3.1. There is such a level set for each odd dimension. For example, $\nu^{-1}(0) \subset \mathbb{H}P^4$ is diffeomorphic to a sphere S^7 , yielding as reduced manifold $\mathbb{H}P^1$ and fibering in circles over its twistor space $\mathbb{C}P^3$, a Kähler submanifold of the Grassmannian $Gr_2(\mathbb{C}^5)$. Thus Sasakian (η) -Einstein submanifolds of type S^5 and $S^3 \times S^2$ can be determined in S^7 (cf. [31]), and intersection properties like in Proposition 5.1 are obtained.

The 11-dimensional example is the level set $\nu^{-1}(0) \subset \mathbb{H}P^5$, identified in Theorem 3.2 with an intersection of three focal sets in $\mathbb{H}P^5$. This 11-dimensional manifold is diffeomorphic to the (unique) 3-Sasakian homogeneous manifold projecting in $SO(3)$ over the Wolf space $\tilde{Gr}_4(\mathbb{R}^6)$, and this latter manifold is isometric to $Gr_2(\mathbb{C}^4)$. Thus again the geometry of 3-Sasakian and of Sasakian (η) -Einstein submanifolds $\nu^{-1}(0) \subset \mathbb{H}P^4$ is obtained from the same starting point as Proposition 5.1.

Besides this kind of geometry of submanifolds, the Sasakian-Einstein level sets $\mu^{-1}(0)$ share with some of the 3-Sasakian level sets $\nu^{-1}(0)$ a common expression of their Poincaré polynomials. We have in fact:

Proposition 5.2. *The Poincaré polynomial of $\mu^{-1}(0) \subset \mathbb{H}P^n$ is given by:*

$$Poin_{\mu^{-1}(0)}(t) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (t^{4i} + t^{4n-3-4i})$$

This is obtained by the Gysin sequence of the S^1 -bundle $\mu^{-1}(0) \rightarrow Gr_2(\mathbb{C}^{n+1})$, where the connecting homomorphism

$$H^p(Gr_2(\mathbb{C}^{n+1})) \rightarrow H^{p+2}(Gr_2(\mathbb{C}^{n+1}))$$

is given by the wedge product with the Kähler form of the Grassmannian. Since this wedge product is injective up to the middle real dimension $p + 2 = 2n - 2$ (cf. [22], lemma 3.1, for the similar 3-Sasakian situation), the Gysin sequence reduces to a series of short exact sequences finishing with

$$H^{2n-4}(Gr_2(\mathbb{C}^{n+1})) \rightarrow H^{2n-2}(Gr_2(\mathbb{C}^{n+1})) \rightarrow H^{2n-2}(\mu^{-1}(0)).$$

This allows to compute the Betti numbers of $(\mu^{-1}(0))$ by differences of consecutive even Betti numbers in $Gr_2(\mathbb{C}^{n+1})$. The Poincaré polynomial of $Gr_2(\mathbb{C}^{n+1})$ is well known (see, for example, [9], p. 292) and by writing it as:

$$(5.1) \quad Poin_{Gr_2(\mathbb{C}^{n+1})} = (1 + t^2 + \dots + t^{2n-2})(1 + t^4 + t^{4m-4}), \quad (n+1 = 2m)$$

$$(5.2) \quad Poin_{Gr_2(\mathbb{C}^{n+1})} = (1 + t^4 + \dots + t^{4m-4})(1 + t^2 + t^{2n}), \quad (n+1 = 2m+1)$$

the conclusion is easily obtained.

The following table gives the Poincaré polynomial of $\mu^{-1}(0) \subset \mathbb{H}P^n$ for low values of n

$n = 3$	$1 + t^4 + t^5 + t^9$
$n = 4$	$1 + t^4 + t^9 + t^{13}$
$n = 5$	$1 + t^4 + t^8 + t^9 + t^{13} + t^{17}$
$n = 6$	$1 + t^4 + t^8 + t^{13} + t^{17} + t^{21}$
$n = 7$	$1 + t^4 + t^8 + t^{12} + t^{13} + t^{17} + t^{21} + t^{25}$
$n = 8$	$1 + t^4 + t^8 + t^{12} + t^{17} + t^{21} + t^{25} + t^{29}$

The Poincaré polynomial of $\mu^{-1}(0)$ can be compared with that of $\nu^{-1}(0)$, computed as for the homogeneous 3-Sasakian manifold $SO(n+1)/(SO(n-3) \times Sp(1))$. The latter has two different expressions, according to whether $n+1$ is even or odd (see [22] or [11], p. 28). For odd values of $n+1 = 2k+3$ this expression is:

$$Poin_{\nu^{-1}(0)}(t) = \sum_{i=0}^{k-1} (t^{4i} + t^{8k-1-4i}),$$

and taking account of the dimensions, this is the same formula given in Proposition 5.2 for $Poin_{\mu^{-1}(0)}(t)$.

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